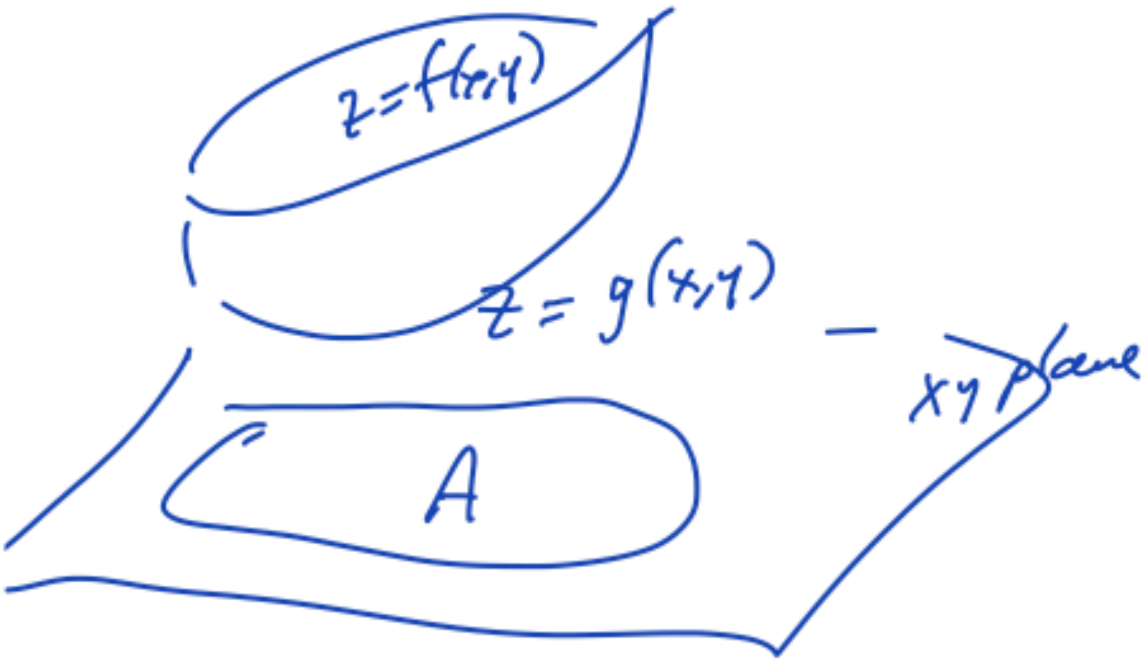
Volume between $z = f(x,y)$ & $z = g(x,y)$



$$\text{Volume} = \iint_A [\underbrace{f(x,y)}_{\text{upper}} - \underbrace{g(x,y)}_{\text{lower}}]\, dA$$

$$\text{Volume} = \iint_A \int_{z=g(x,y)}^{z=f(x,y)} 1 \, dz \, dA$$

$r\,dr\,d\theta$

$dx\,dy$

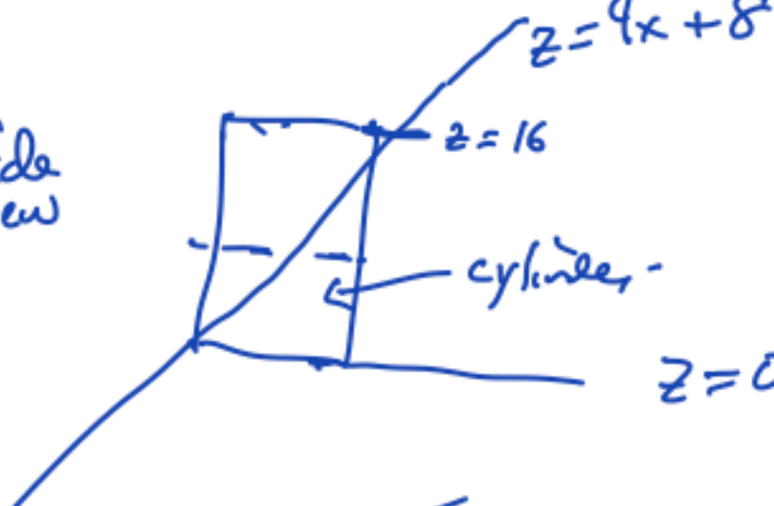
$dy\,dx$

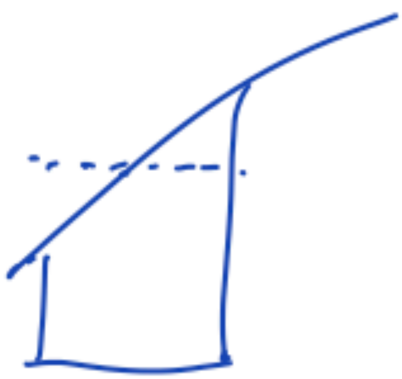
(3.4) $S = \{(x,y,z) : \underline{\underline{x^2+y^2 \leq 4}},\ 0 \leq z \leq 4x+8\}$

$z \geq 0$ plane
xy plane

$z = 4x+8$ plane

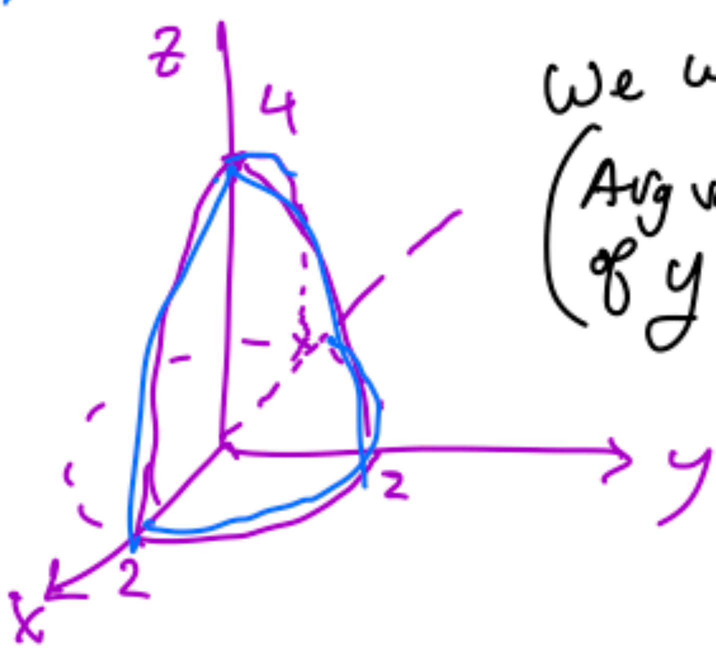
side view





Calculation from last time:

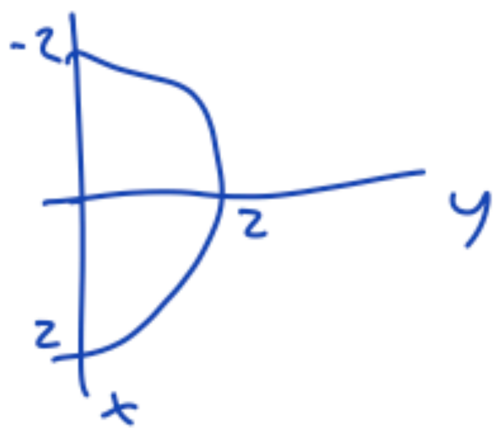
<u>Example</u>: Let Ω be the region between the surface $z = 4 - x^2 - y^2$, the xy plane, and the xz plane in the part where $z > 0$, $y > 0$. Find the average value of y in this region.

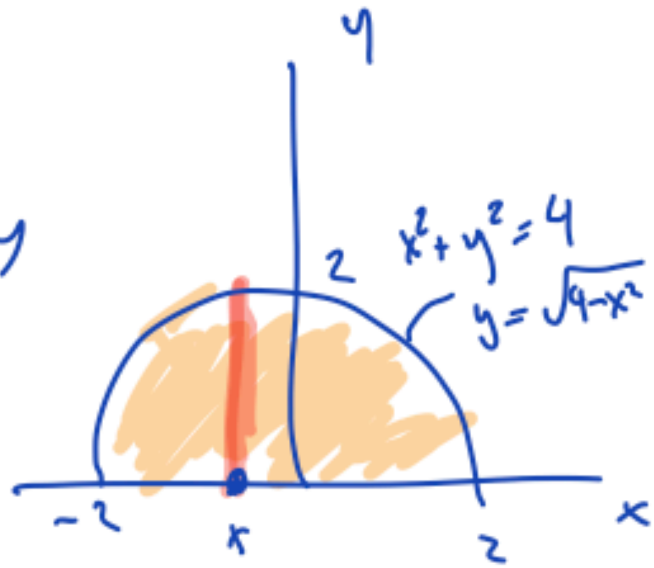


We want

$$\begin{pmatrix}\text{Avg value}\\ \text{of } y\end{pmatrix} = \frac{\iiint_\Omega y\,dx\,dy\,dz}{\iiint_\Omega 1\,dx\,dy\,dz}$$

$$\underbrace{\iint}_{\substack{\text{region} \\ \text{in xy plane}}} \int_{\substack{z = \text{lower surface} \\ z=0}}^{z = \text{upper surface} = 4-x^2-y^2} y \quad \underline{dV}$$





vertical slices

$$0 \le y \le \sqrt{4-x^2}$$

$$-2 \le x \le 2$$

$$\iiint y \, dV = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} \int_{z=0}^{z=4-x^2-y^2} y \; dz \, dy \, dx$$

$$= \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} yz \Big|_{0}^{4-x^2-y^2} dy \; dx$$

$$= \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} (4y - x^2y - y^3)\,dy\,dx$$

$$= \int_{-2}^{2} 2y^2 - \frac{x^2y^2}{2} - \frac{y^4}{4} \Big|_{y=0}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 2(4-x^2) - \frac{x^2(4-x^2)}{2} - \frac{(4-x^2)^2}{4}\,dx$$

$$= \int_{-2}^{2} 8 - 2x^2 - 2x^2 + \frac{x^4}{2} - \frac{(16 - 8x^2 + x^4)}{4}\,dx$$

$$-4 + 2x^2 - \frac{x^4}{4}$$

$$= \int_{-2}^{2} 4 - 2x^2 + \frac{x^4}{4}\,dx$$

$$= 4x - \frac{2}{3}x^3 + \frac{1}{20}x^5 \Big|_{-2}^{2}$$

$$= 16 - \frac{32}{3} + \cancel{\frac{64}{20}} \quad \frac{16}{5} = \frac{240 - 160 + 48}{15}$$

$$= \boxed{\frac{128}{15}}.$$

Denominator:

$$\iiint 1 \, dV$$

$$= \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} \int_{z=0}^{z=4-x^2-y^2} 1 \, dz \, dy \, dx$$

$$= \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} 4-x^2-y^2 \, dy \, dx$$

$$= \int_{-2}^{2} 4y - x^2 y - \frac{y^3}{3} \Big|_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 4\sqrt{4-x^2} - x^2\sqrt{4-x^2} - \frac{(4-x^2)^{3/2}}{3} \, dx$$

$$= \int_{-2}^{2} \frac{2(4-x^2)^{3/2}}{3}\, dx$$

$$= \frac{2}{3} \int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

$$= 4\pi \qquad \text{(see last class calculation)}$$

Avg value: $\dfrac{128/15}{4\pi} = \cancel{\boxed{\dfrac{7}{15\pi}}}\,.$

$$\boxed{\frac{32}{15\pi}}$$

Different approach — let's use polar coordinates ("cylindrical coordinates" r, θ, z).

xy plane

$0 \le z \le 4 - x^2 - y^2$
$= 4 - r^2$

$0 \le r \le 2$

$0 \le \theta \le \pi$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\iiint y\,dV = \int_{\theta=0}^{\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} r\sin\theta \; r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{2} \left(r^2\sin\theta\right) z \Big|_{z=0}^{(4-r^2)} dr\,d\theta$$

$$= \int_0^{\pi} \int_{r=0}^{2} \left(4r^2 - r^4\right) \sin\theta \; dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \left(\frac{4}{3}r^3 - \frac{r^5}{5}\right)\Big|_0^2 \; d\theta$$

$$= \int_0^{\pi} \sin\theta \left(\frac{32}{3} - \frac{32}{5}\right) d\theta$$

$$\frac{160 - 96}{15} \qquad \frac{64}{15}$$

$$= \frac{64}{15} \left(-\cos\theta \Big|_0^{\pi}\right)$$

$$= \frac{64}{15} \left(-\underset{-1}{\cos\pi} + \underset{1}{\cos(0)}\right) = \boxed{\frac{128}{15}}.$$

$$\iiint 1 \, dV = \int_{\theta=0}^{\pi} \int_{r=0}^{2} \int_{z=0}^{4-r^2} 1 \, r \, dz \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^{2} rz \Big|_0^{4-r^2} dr \, d\theta$$

$$= \int_0^{\pi} \int_0^{2} 4r - r^3 \, dr \, d\theta$$

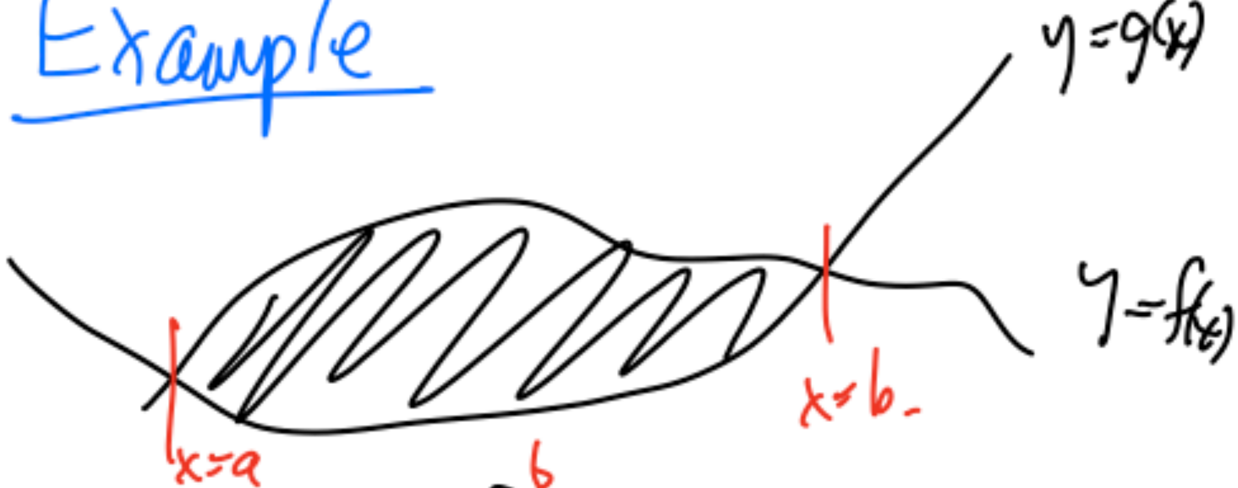
$$= \int_0^{\pi} \left(2r^2 - \frac{r^4}{4} \Big|_0^2 \right) d\theta$$

$$\underbrace{8 - \frac{16}{4}}_{4}$$

$$= \int_0^{\pi} 4 \, d\theta = \boxed{4\pi}.$$

Avg value of y $= \dfrac{128/15}{4\pi} = \boxed{\dfrac{32}{15\pi}}.$

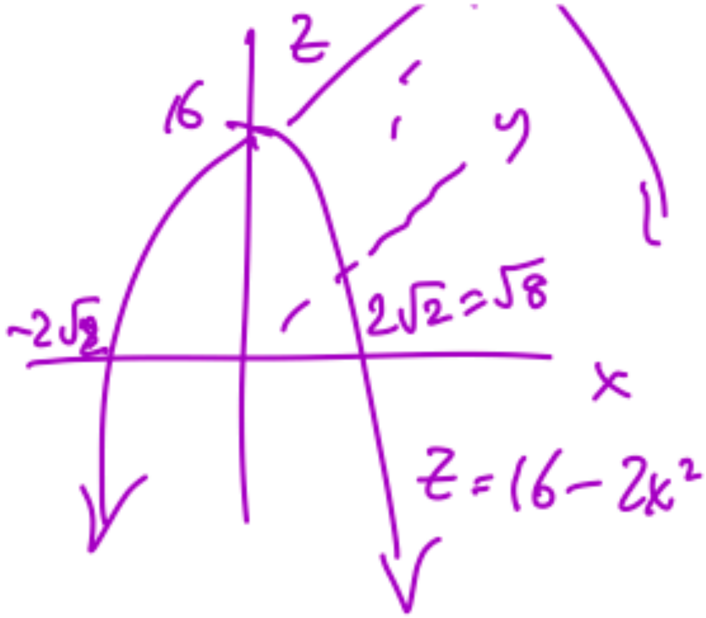
Example



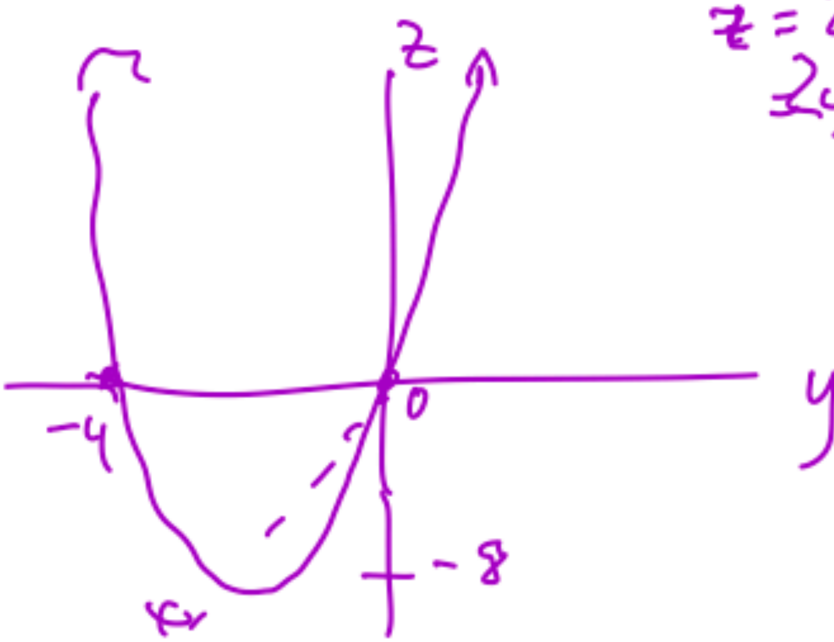
$$\text{Area} = \int_a^b \left(f(x) - g(x)\right)\,dx$$

Higher-D example.

Find the volume between the surfaces $z = 16 - 2x^2$ (1) and $z = 2y^2 + 8y$ (2)



$z = 2y^2 + 8y$
$= 2y(y+4)$



Intersection of 2 surfaces

$z = \underbrace{2y^2 + 8y}_{\text{bottom}} = 16 - 2x^2$

restricted to xy plane

$$2y^2+8y+2x^2=16$$

$$y^2+4y+x^2=8$$

$$+4 \qquad\qquad +4$$

$$(y+2)^2+x^2=12$$

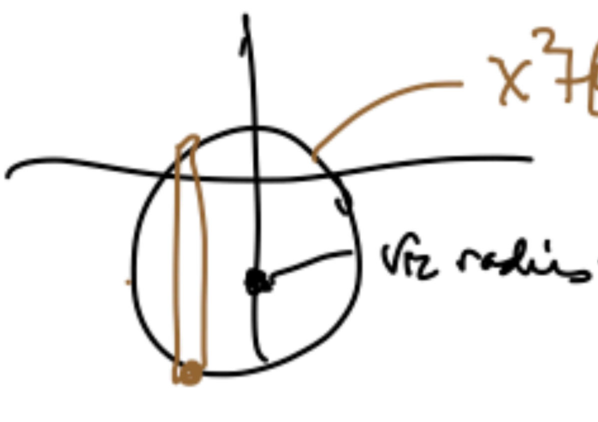
circle centered at $(0,-2)$

radius $\sqrt{12}$

So we will integrate over this disk.

$$\text{Volume} = \iint_{\text{disk}} (\text{upper } z)-(\text{lower } z)\, dA$$

$$= \iint_{\text{disk}} \left(16-2x^2-(2y^2+8y)\right) dA$$



$$x^2+(y+2)^2=12$$

$$(y+2)^2=12-x^2$$

$$y+2=\pm\sqrt{12-x^2}$$

$$y = -2 \pm \sqrt{12 - x^2}$$

$$-\sqrt{12} \le x \le \sqrt{12}$$

$\Rightarrow$ Volume

$$= \int_{x=-\sqrt{12}}^{\sqrt{12}} \int_{y=-2-\sqrt{12-x^2}}^{-2+\sqrt{12-x^2}} (16 - 2x^2 - 2y^2 - 8y)\, dy\, dx$$

$=$ sagemath - ...

Could also do a version of polar coordinates centered at $(0, -2)$

$$x = r\cos\theta$$
$$y = r\sin\theta - 2$$
$$dy\,dx = r\,dr\,d\theta$$

$0 \le r \le \sqrt{12}$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \int_0^{\sqrt{2}} \int_{\theta=0}^{2\pi} \left(\left(16 - 2r^2\cos^2\theta - 2(r\sin\theta - 2)\right)^2 - 8(r\sin\theta - 2) \right) r \, d\theta \, dr$$

$$= \ldots\ldots = \text{answer}$$